Miscellaneous Examples

Example 41 If R_1 and R_2 are equivalence relations in a set A, show that $R_1 \cap R_2$ is also an equivalence relation.

Solution Since R_1 and R_2 are equivalence relations, $(a, a) \in R_1$, and $(a, a) \in R_2$ $\forall a \in A$. This implies that $(a, a) \in R_1 \cap R_2$, $\forall a$, showing $R_1 \cap R_2$ is reflexive. Further, $(a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2 \Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2 \Rightarrow (b, a) \in R_1 \cap R_2$, hence, $R_1 \cap R_2$ is symmetric. Similarly, $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2 \Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2 \Rightarrow (a, c) \in R_1 \cap R_2$. This shows that $R_1 \cap R_2$ is transitive. Thus, $R_1 \cap R_2$ is an equivalence relation.

Example 42 Let R be a relation on the set A of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if $xv = yu$. Show that R is an equivalence relation.

Solution Clearly, (x, y) R (x, y), $\forall (x, y) \in A$, since $xy = yx$. This shows that R is reflexive. Further, (x, y) R $(u, v) \Rightarrow xv = yu \Rightarrow uy = vx$ and hence (u, v) R (x, y). This shows that R is symmetric. Similarly, (x, y) R (u, v) and (u, v) R $(a, b) \Rightarrow xv = yu$ and $ub = va \Rightarrow xv\frac{a}{u} = yu\frac{a}{u} \Rightarrow xv\frac{b}{v} = yu\frac{a}{u} \Rightarrow xb = ya$ and hence (x, y) R (a, b). Thus, R is transitive. Thus, R is an equivalence relation.

Example 43 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let R_1 be a relation in X given by $R_1 = \{(x, y) : x - y \text{ is divisible by } 3\}$ and R_2 be another relation on X given by $R_2 = \{(x, y): \{x, y\} \subset \{1, 4, 7\}\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}\}$. Show that $R_1 = R_2$.

Solution Note that the characteristic of sets {1, 4, 7}, {2, 5, 8} and {3, 6, 9} is that difference between any two elements of these sets is a multiple of 3. Therefore, $(x, y) \in R_1 \Rightarrow x - y$ is a multiple of $3 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow (x, y) \in R_2$. Hence, $R_1 \subset R_2$. Similarly, $\{x, y\} \in R_2 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow x - y$ is divisible by $3 \Rightarrow \{x, y\} \in R_1$. This shows that $R_2 \subset R_1$. Hence, $R_1 = R_2$.

Example 44 Let $f : X \to Y$ be a function. Define a relation R in X given by $R = \{(a, b): f(a) = f(b)\}$. Examine whether R is an equivalence relation or not.

Solution For every $a \in X$, $(a, a) \in R$, since $f(a) = f(a)$, showing that R is reflexive. Similarly, $(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$. Therefore, R is symmetric. Further, $(a, b) \in R$ and $(b, c) \in R \Rightarrow f(a) = f(b)$ and $f(b) = f(c) \Rightarrow f(a) = f(c) \Rightarrow (a, c) \in R$, which implies that R is transitive. Hence, R is an equivalence relation.

Example 45 Determine which of the following binary operations on the set R are associative and which are commutative.

(a) $a * b = 1 \ \forall \ a, b \in R$ (b) $a * b = \dfrac{(a+b)}{2} \ \forall \ a, b \in R$

Solution

(a) Clearly, by definition $a * b = b * a = 1$, $\forall a, b \in R$. Also $(a * b) * c = (1 * c) = 1$ and $a * (b * c) = a * (1) = 1$, $\forall \ a, b, c \in R$. Hence R is both associative and commutative.

(b) $a * b = \dfrac{a+b}{2} = \dfrac{b+a}{2} = b * a$, shows that $*$ is commutative. Further,

$$(a * b) * c = \left(\frac{a+b}{2}\right) * c.$$

$$= \frac{\left(\frac{a+b}{2}\right) + c}{2} = \frac{a+b+2c}{4}.$$

But

$$a * (b * c) = a * \left(\frac{b+c}{2}\right)$$

$$= \frac{a + \frac{b+c}{2}}{2} = \frac{2a+b+c}{4} \neq \frac{a+b+2c}{4} \text{ in general.}$$

Hence, $*$ is not associative.

Example 46 Find the number of all one-one functions from set A = {1, 2, 3} to itself.

Solution One-one function from {1, 2, 3} to itself is simply a permutation on three symbols 1, 2, 3. Therefore, total number of one-one maps from {1, 2, 3} to itself is same as total number of permutations on three symbols 1, 2, 3 which is 3! = 6.

Example 47 Let A = {1, 2, 3}. Then show that the number of relations containing (1, 2) and (2, 3) which are reflexive and transitive but not symmetric is three.

Solution The smallest relation R_1 containing (1, 2) and (2, 3) which is reflexive and transitive but not symmetric is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}. Now, if we add the pair (2, 1) to R_1 to get R_2, then the relation R_2 will be reflexive, transitive but not symmetric. Similarly, we can obtain R_3 by adding (3, 2) to R_1 to get the desired relation. However, we can not add two pairs (2, 1), (3, 2) or single pair (3, 1) to R_1 at a time, as by doing so, we will be forced to add the remaining pair in order to maintain transitivity and in the process, the relation will become symmetric also which is not required. Thus, the total number of desired relations is three.

Example 48 Show that the number of equivalence relation in the set {1, 2, 3} containing (1, 2) and (2, 1) is two.

Solution The smallest equivalence relation R_1 containing (1, 2) and (2, 1) is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}. Now we are left with only 4 pairs namely (2, 3), (3, 2), (1, 3) and (3, 1). If we add any one, say (2, 3) to R_1, then for symmetry we must add (3, 2) also and now for transitivity we are forced to add (1, 3) and (3, 1). Thus, the only equivalence relation bigger than R_1 is the universal relation. This shows that the total number of equivalence relations containing (1, 2) and (2, 1) is two.

Example 49 Show that the number of binary operations on {1, 2} having 1 as identity and having 2 as the inverse of 2 is exactly one.

Solution A binary operation $*$ on {1, 2} is a function from {1, 2} × {1, 2} to {1, 2}, i.e., a function from {(1, 1), (1, 2), (2, 1), (2, 2)} → {1, 2}. Since 1 is the identity for the desired binary operation $*$, $*(1, 1) = 1$, $*(1, 2) = 2$, $*(2, 1) = 2$ and the only choice left is for the pair (2, 2). Since 2 is the inverse of 2, i.e., $*(2, 2)$ must be equal to 1. Thus, the number of desired binary operation is only one.

Example 50 Consider the identity function $I_N : \mathbf{N} \to \mathbf{N}$ defined as $I_N(x) = x \ \forall x \in \mathbf{N}$. Show that although I_N is onto but $I_N + I_N : \mathbf{N} \to \mathbf{N}$ defined as

$$(I_N + I_N)(x) = I_N(x) + I_N(x) = x + x = 2x$$ is not onto.

Solution Clearly I_N is onto. But $I_N + I_N$ is not onto, as we can find an element 3 in the co-domain **N** such that there does not exist any x in the domain **N** with $(I_N + I_N)(x) = 2x = 3$.

Example 51 Consider a function $f : \left[0, \frac{\pi}{2}\right] \rightarrow \mathbf{R}$ given by $f(x) = \sin x$ and $g : \left[0, \frac{\pi}{2}\right] \rightarrow \mathbf{R}$ given by $g(x) = \cos x$. Show that f and g are one-one, but $f + g$ is not one-one.

Solution Since for any two distinct elements x_1 and x_2 in $\left[0, \frac{\pi}{2}\right]$, $\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$, both f and g must be one-one. But $(f + g)(0) = \sin 0 + \cos 0 = 1$ and $(f + g)\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$. Therefore, $f + g$ is not one-one.

Miscellaneous Exercise on Chapter 1

Summary

In this chapter, we studied different types of relations and equivalence relation, composition of functions, invertible functions and binary operations. The main features of this chapter are as follows:

- *Empty relation* is the relation R in X given by $R = \phi \subset X \times X$.
- *Universal relation* is the relation R in X given by $R = X \times X$.
- *Reflexive relation* R in X is a relation with $(a, a) \in R \ \forall a \in X$.
- *Symmetric relation* R in X is a relation satisfying $(a, b) \in R$ implies $(b, a) \in R$.
- *Transitive relation* R in X is a relation satisfying $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$.
- *Equivalence relation* R in X is a relation which is reflexive, symmetric and transitive.
- *Equivalence class* $[a]$ containing $a \in X$ for an equivalence relation R in X is the subset of X containing all elements b related to a.
- A function $f : X \rightarrow Y$ is *one-one* (or *injective*) if

  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$.
- A function $f : X \rightarrow Y$ is *onto* (or *surjective*) if given any $y \in Y, \exists x \in X$ such that $f(x) = y$.
- A function $f : X \rightarrow Y$ is *one-one and onto* (or *bijective*), if f is both one-one and onto.
- The *composition* of functions $f : A \rightarrow B$ and $g : B \rightarrow C$ is the function $gof : A \rightarrow C$ given by $gof(x) = g(f(x)) \forall \ x \in A$.
- A function $f : X \rightarrow Y$ is *invertible* if $\exists \ g : Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$.
- A function $f : X \rightarrow Y$ is *invertible* if and only if f is one-one and onto.

- Given a finite set X, a function $f : X \to X$ is one-one (respectively onto) if and only if f is onto (respectively one-one). This is the characteristic property of a finite set. This is not true for infinite set
- A ***binary operation*** $*$ on a set A is a function $*$ from $A \times A$ to A.
- An element $e \in X$ is the ***identity*** element for binary operation $* : X \times X \to X$, if $a * e = a = e * a \ \forall a \in X$.
- An element $a \in X$ is ***invertible*** for binary operation $* : X \times X \to X$, if there exists $b \in X$ such that $a * b = e = b * a$ where, e is the identity for the binary operation $*$. The element b is called ***inverse*** of a and is denoted by a^{-1}.
- An operation $*$ on X is ***commutative*** if $a * b = b * a \ \forall a, b$ in X.
- An operation $*$ on X is ***associative*** if $(a * b) * c = a * (b * c) \forall a, b, c$ in X.